13.1: Intro to 3D Vector Curves

To visualize 3D-curves, we start by

Step 1: Find surface/path of motion. ← eliminate parameter

Step 2: Plot points.

$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$

Probably motion on circle or ellipse

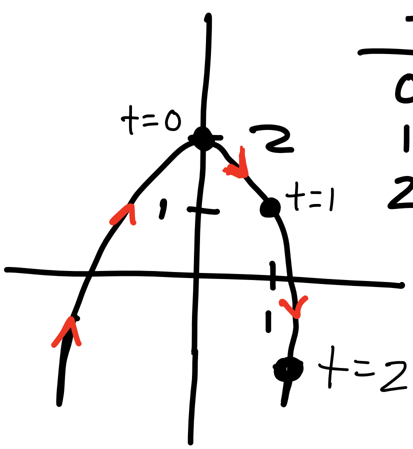
2D Examples

Eliminate the parameters

1. $x = t, y = 2 - t^2$ motion

not the same

$y = 2 - x^2 \rightarrow$ path



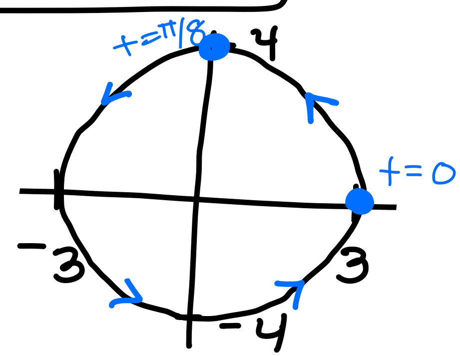
t	x	y
0	0	2
1	1	1
2	2	-2

2. $x = 3\cos(4t), y = 4\sin(4t)$

$\cos(4t) = \frac{x}{3}$ $\sin(4t) = \frac{y}{4}$

$(\cos(4t))^2 + (\sin(4t))^2 = 1$

$\frac{x^2}{9} + \frac{y^2}{16} = 1 \rightarrow$ ellipse!



t	x	y
0	3	0
$\frac{\pi}{8}$	0	4

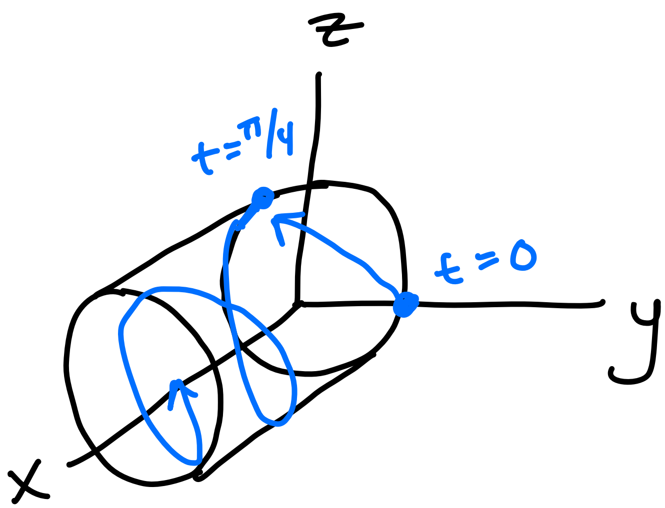
3D Example

$x = t, y = \cos(2t), z = \sin(2t)$

$y = \cos(2x)$ ← cosine cylinder

$z = \sin(2x)$ ← sine cylinder

$y^2 + z^2 = 1$ ← circular cylinder

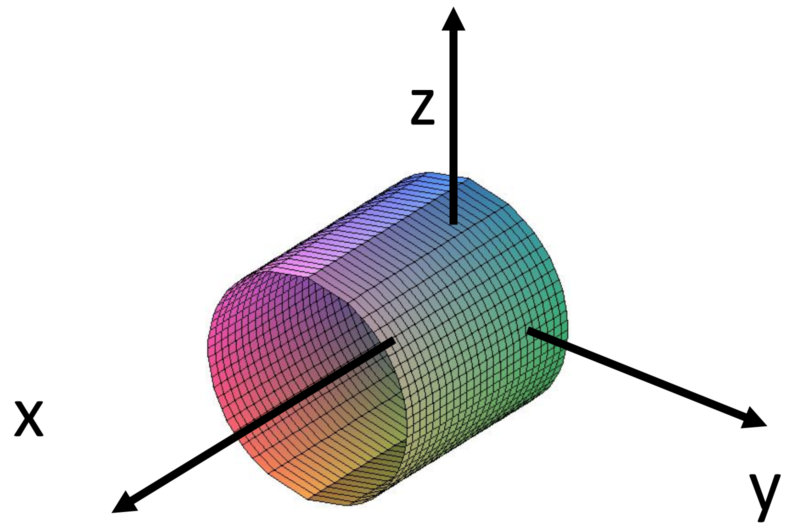


t	x	y	z
0	0	1	0
π/4	π/4	0	1

Example: All pts given by the equations

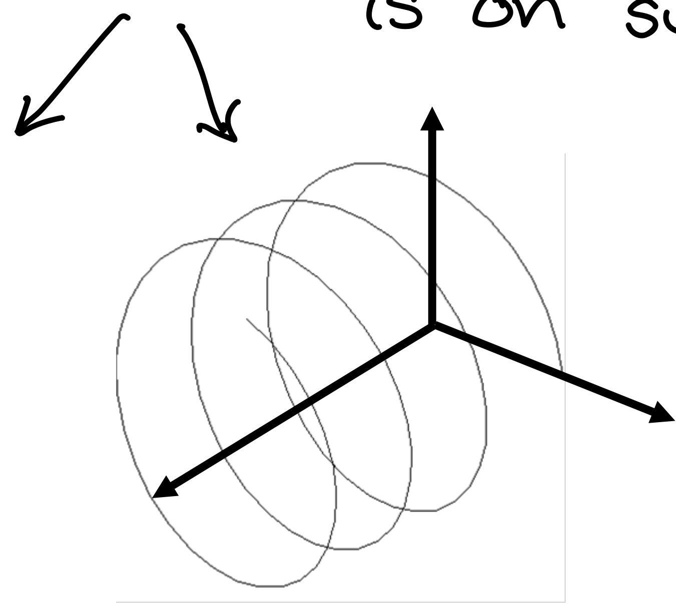
$$x = t\ , y = \cos(2t)\ \ , z = \sin(2t)$$

are on the cylinder*:* $y^2 + z^2 = 1.$



Now plot points!

not the same, but every point on curve is on surface



next note a few pages down

⇓

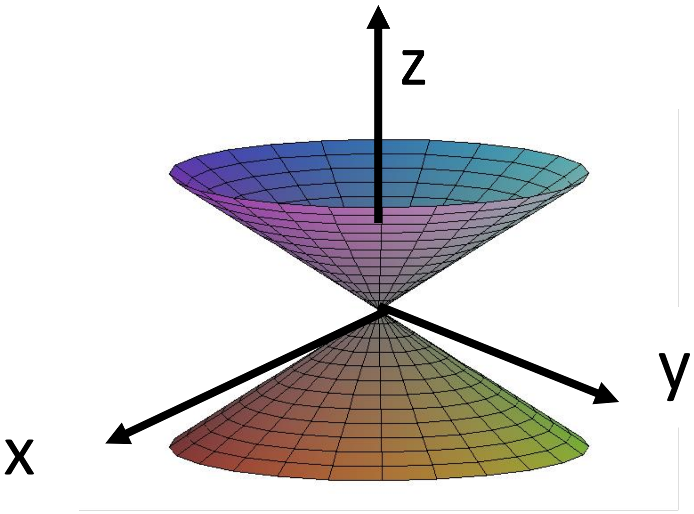
Another 3D Examples

$$x = t\cos(t), y = t\sin(t)\,, z = t$$

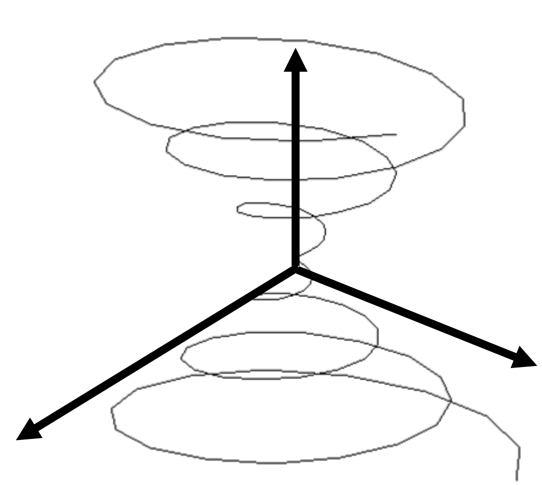
Example: All pts given by the equations

$$x = t\cos(t), y = t\sin(t), z = t$$

are on the cone $z^2 = x^2 + y^2$.



Now plot points!



Intersection issues

For all intersection questions, combine the conditions

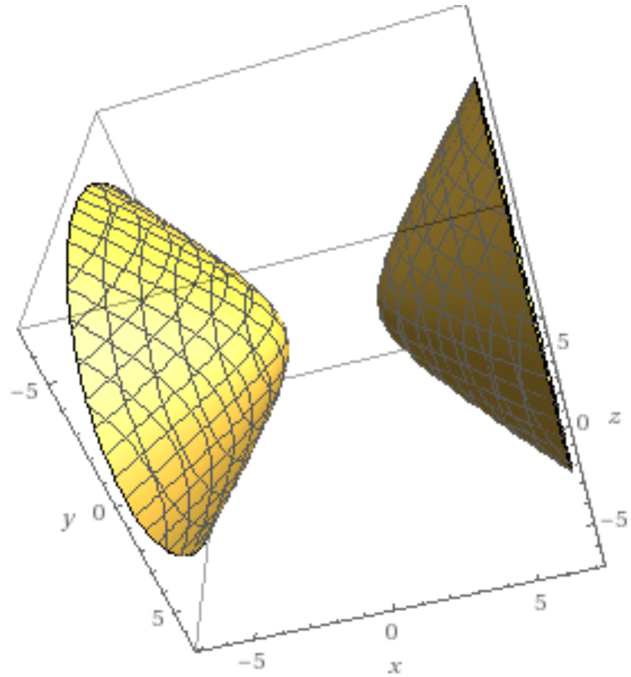
(a) ***Intersecting a curve and surface***.
Combine conditions

Example:
Find all intersections of

$$x = t, y = \cos(\pi\, t)\,, z = \sin(\pi\, t)$$

with the surface

$$x^2 - y^2 - z^2 = 3.$$



(b) ***Intersecting two curves.***

Use two different parameters!!!

Combine conditions.

We say the objects **collide** if the intersection happens at the same parameter value (i.e. same time).

Example:

Two particles are moving according to

$$\boldsymbol{r}_1(t) = \langle t, 5t, 9\rangle, \text{and}$$
$$\boldsymbol{r}_2(t) = \langle t - 2, 5, t^2\rangle.$$

Do their paths intersect?

Do they collide?

(c) ***Intersecting two surfaces***.

Answer will be a 3D curve.

To parameterize the curve:

Let one variable be t. Solve for others in terms of t.

OR

For circle/ellipse try

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Leftrightarrow \begin{matrix} x = \mathrm{a}\cos(t) \\ y = \mathrm{b}\sin(t) \end{matrix}$$

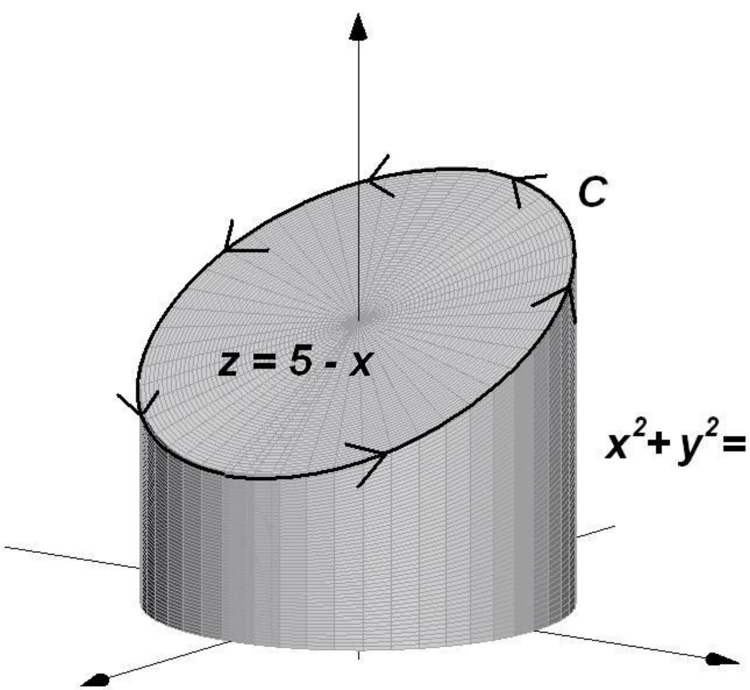
Examples

1. Find *any* parametric equations that describe the curve of intersection of

$$z = 2x + y^2 \text{ and } z = 2y$$

2. Find *any* parametric equations that describe the curve of intersection of

$$x^2 + y^2 = 1 \text{ and } z = 5 - x$$



3. Find *any* parametric equations that describe the curves of intersection of
$x^2 + y^2 + z^2 = 1$ and $z^2 = x^2 + y^2$

13.1: Intro to 3D Vector Curve

Entry Task: (HW 13.1/4)

Given

$$x = t\cos(t), y = t, z = t\sin(t)$$

$x = y\cos(y)$ $z = y\sin(y)$

$(\cos(y))^2 + (\sin(y))^2 = 1$

$\cos(y) = \frac{x}{y}$ $\sin(y) = \frac{z}{y}$

$\left(\frac{x^2}{y^2} + \frac{z^2}{y^2} = 1\right) y^2$

$x^2 + z^2 = y^2$

$x^2 + z^2 - y^2 = 0$

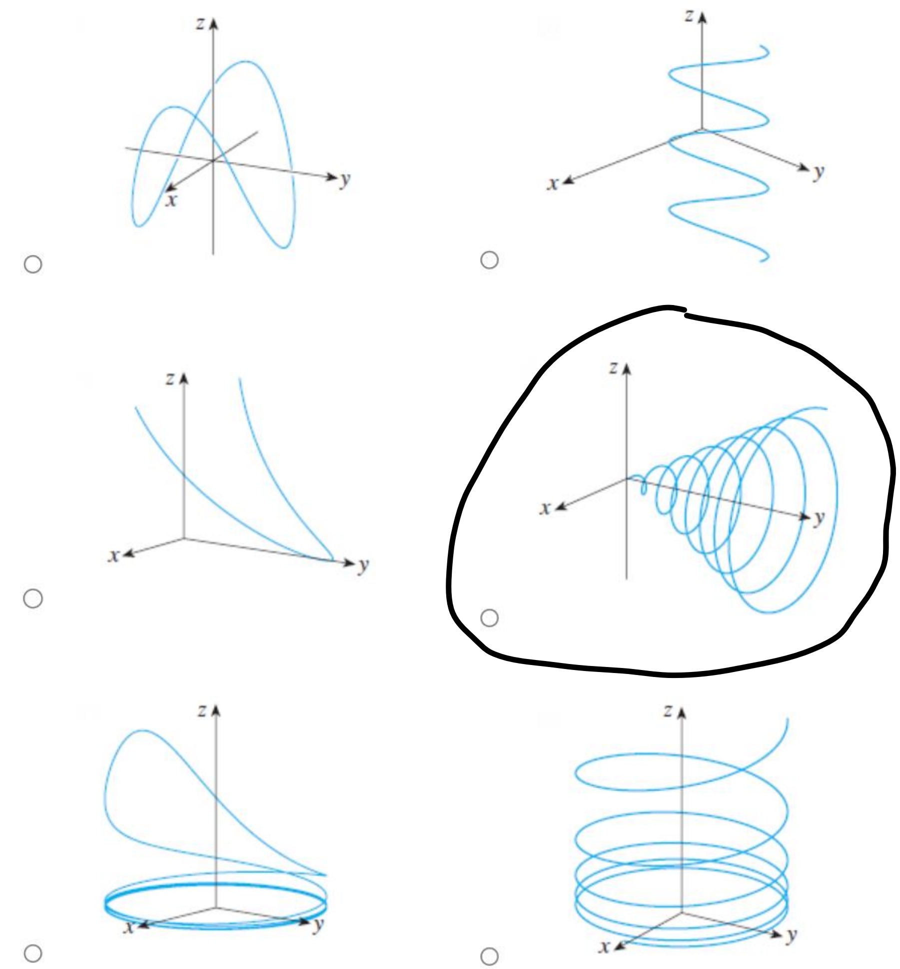
cone!

Eliminate the parameter until you get a

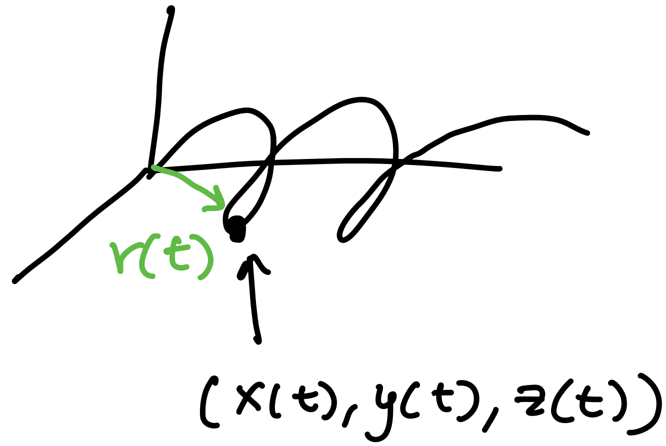
4. Question Details

Match the parametric equations with the correct graph.

$x = t\cos(t)$, $y = t$, $z = t\sin(t)$, $t \geq 0$



A couple side notes:



Forms

Parametric Form

$$x = t\cos(t), y = t, z = t\sin(t)$$

Position Vector Form

$$\boldsymbol{r}(t) = \langle t\cos(t), t, t\sin(t)\rangle$$

Limits

do 3 seperate limits

6. Question Details

Find the limit.

$$\lim_{t \to 0} \left(e^{-6t}\,\mathbf{i} + \frac{t^2}{\sin^2(t)}\,\mathbf{j} + \sin(2t)\,\mathbf{k} \right)$$

type in vector form

<1, ?, 0>

$$\lim_{t \to 0} e^{-6t} = 1$$

$$\lim_{t \to 0} \frac{t^2}{\sin^2(t)} \quad \frac{0}{0} \quad = \text{look it up...}$$

$$\lim_{t \to 0} \sin(2t) = 0$$

<u>Three types of intersections</u>

General principle, combine conditions!

(a) ***("Easy") A curve and surface***.
Combine conditions, solve for *t.*

(b) ***("Medium") Two curves***.
Use different parameters!
Combine conditions, solve for both parameters.

(c) ***("Hard?")*** **Two surfaces.**
These can be tricky.
Answer will be a *curve*!
A typical goal is to try to *parameterize* the curve, here's now…

Combine conditions:

- Pick one variable as t, solve for others.
- And/or use circular motion.

Examples: *(HW 13.1 / 7, 8)*

Find the intersection of

$$x = t, y = \cos(\pi t), z = \sin(\pi t)$$

and

$$x^2 - y^2 - z^2 = 3.$$

Example: *(HW 13.1/12, 13)*

Sp'17 Exam Problem

Given:

$$\boldsymbol{r}_1(t) = \langle 2t, 3t^2, 2t^3 \rangle$$

$$\boldsymbol{r}_2(t) = \langle 2 - 2t, 3 + 3t, 2 - 6t \rangle$$

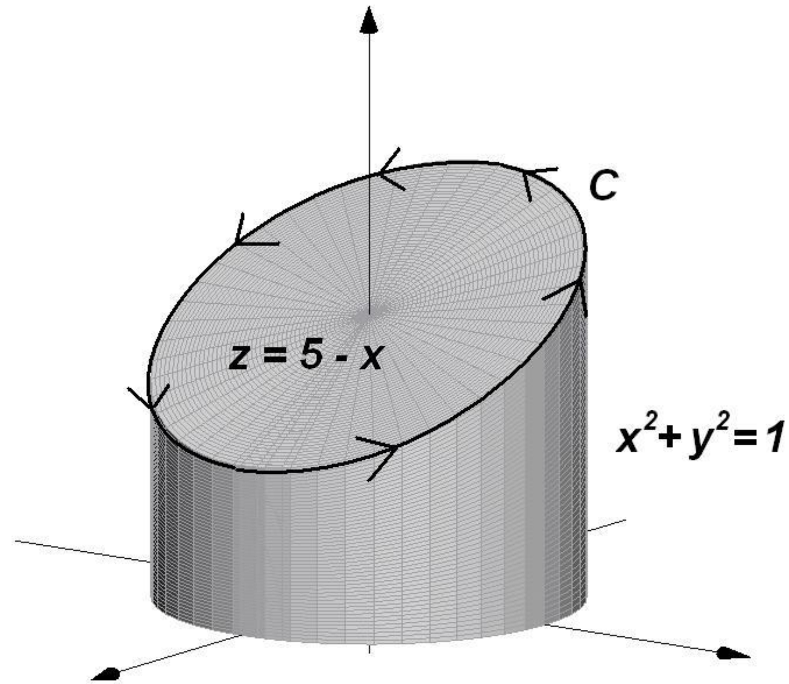
Find the (x,y,z) point(s) at which the **paths** of the two particles described cross.

Visuals: https://www.math3d.org/NKOiXjHB

Example: *(HW 13.1/9-11)*

Find *any* parametric equations that describe the curve of intersection of

$$x^2 + y^2 = 1 \text{ and } z = 5 - x$$



Visuals: https://www.math3d.org/ickrjljF

9. Question Details

Find a vector function, $\mathbf{r}(t)$, that represents the curve of intersection of the two surfaces.

The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$

$\mathbf{r}(t) =$

10. Question Details

Find a vector function, $\mathbf{r}(t)$, that represents the curve of intersection of the two surfaces.

The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2 + y$

$\mathbf{r}(t) =$

11. Question Details

Find a vector function, $\mathbf{r}(t)$, that represents the curve of intersection of the two surfaces.

The paraboloid $z = 2x^2 + y^2$ and the parabolic cylinder $y = 4x^2$

$\mathbf{r}(t) =$

Section 13.2 – Vector Calculus Intro

Do calculus component-wise!

1st Derivative vector:

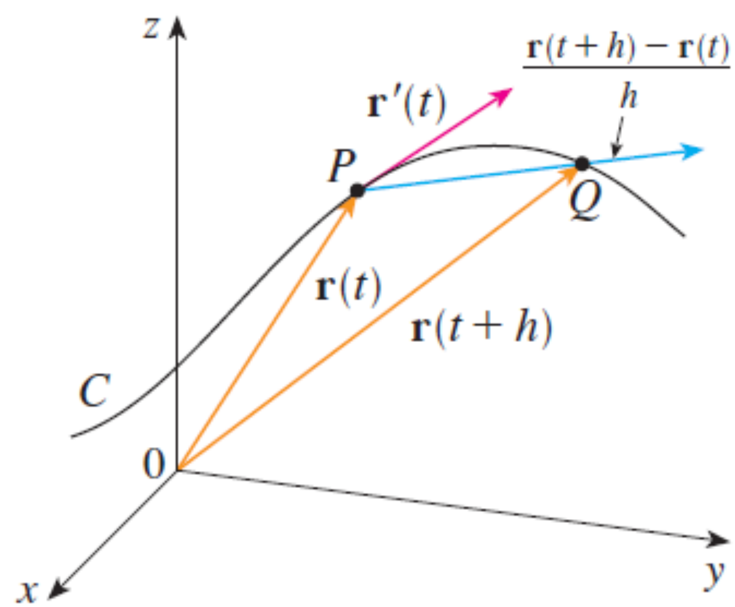
$$\vec{\boldsymbol{r}}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

2nd Derivative vector:

$$\vec{\boldsymbol{r}}''(t) = \langle x''(t), y''(t), z''(t) \rangle$$

Anti-derivative vector:

$$\int \vec{\boldsymbol{r}}(t)dt = \left\langle \int x(t)dt, \int y(t)dt, \int z(t)dt \right\rangle$$



Visuals: https://www.math3d.org/pkXEorK4

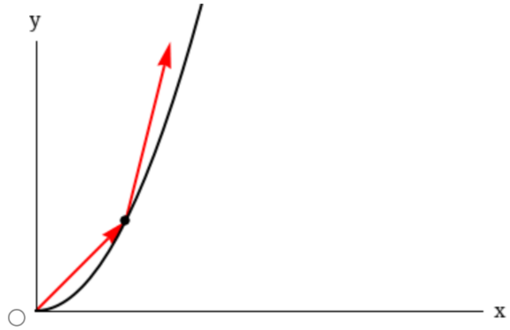
2. Question Details

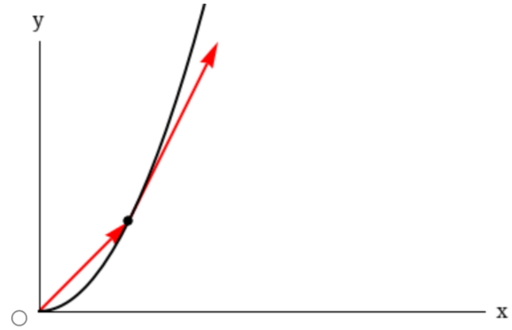
Consider the given vector equation.

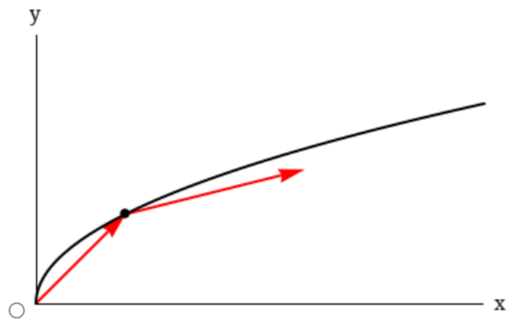
$\mathbf{r}(t) = e^{18t}\,\mathbf{i} + e^{9t}\,\mathbf{j}$

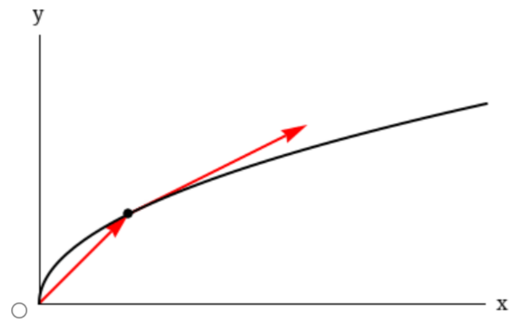
(a) Find $\mathbf{r}'(t)$.

(b) Sketch the plane curve together with position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of $t = 0$.









A couple important follow-up facts

- $\vec{\boldsymbol{T}}(t) = \frac{1}{|\vec{\boldsymbol{r}}'(t)|}\vec{\boldsymbol{r}}'(t)$ *unit tangent vector* (HW 13.2/ 5, 6)

- To find the tangent **line** to $\vec{\boldsymbol{r}}(t)$ at $t = t_0$ (HW 13.2 / 7)
  - *Step 1*: Compute location
    $$\vec{\boldsymbol{r}}(t_0) = \langle x(t_0), y(t_0), z(t_0)\rangle = \langle x_0, y_0, z_0\rangle.$$
  - *Step 2*: Compute a tangent vector
    $$\vec{\boldsymbol{r}}'(t_0) = \langle x'(t_0), y'^{(t_0)}, z'(t_0)\rangle = \langle a, b, c\rangle.$$
  - *Step 3*: $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$

Example: (*HW 13.2 / 7, 8, 9*)

W'15 Exam 1 - Loveless

Consider the curve given by:

$$x = 2t, y = 5, z = t^2 - 10t$$

(a) There is one point on the curve at which the tangent line is parallel to the xy-plane.
Find the tangent line at this point.

(b) Find the angle of intersection of the original curve this other curve

$$x = 7 + u, y = 2u + 11, z = u^2 + u - 22$$

Visuals: https://www.math3d.org/ZgPOXWkh

Derivatives Quick Review

A basic example: Write down the derivative of

$$g(x) = x^3 \cos(2x) + \sqrt{1 + e^{3x}}$$

Calculus Fact Sheet

Essential Derivative Rules

$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(b^x) = b^x \ln(b)$	
$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$
$\frac{d}{dx}(\cos(x)) = -\sin(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$	$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$
$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1}$	$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left(\sec^{-1}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$
$(FS)' = FS' + F'S$	$\left(\frac{N}{D}\right)' = \frac{DN' - ND'}{D^2}$	$[f(g(x))]' = f'(g(x))g'(x)$

From my Calculus 1 Fact Sheet:
math.washington.edu/~aloveles/Math126Materials/CalculusFactSheet2.pdf

Antiderivatives Quick Review

A basic example: Find $f(t)$, if

$$f'(t) = 6\sqrt{t} + \sin(t) + 9te^{t^2}$$

with $f(0) = 7$.

From my Calculus 1 Fact Sheet:
math.washington.edu/~aloveles/Math126Materials/CalculusFactSheet2.pdf

Essential Integral Rules

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$	$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln\lvert ax+b \rvert + C$
$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$	$\int b^x \, dx = \frac{1}{\ln(b)} b^x + C$
$\int \cos(ax) \, dx = \frac{1}{a} \sin(ax) + C$	$\int \sin(ax) \, dx = -\frac{1}{a} \cos(ax) + C$
$\int \sec^2(x) \, dx = \tan(x) + C$	$\int \csc^2(x) \, dx = -\cot(x) + C$
$\int \sec(x)\tan(x) \, dx = \sec(x) + C$	$\int \csc(x)\cot(x) \, dx = -\csc(x) + C$
$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$	$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
$\int \tan(x) \, dx = \ln\lvert \sec(x) \rvert + C$	$\int \cot(x) \, dx = \ln\lvert \sin(x) \rvert + C$
$\int \sec(x) \, dx = \ln\lvert \sec(x) + \tan(x) \rvert + C$	$\int \csc(x) \, dx = \ln\lvert \csc(x) - \cot(x) \rvert + C$

$$\int \sec^3(x) \, dx = \frac{1}{2} \sec(x)\tan(x) + \frac{1}{2} \ln\lvert \sec(x) + \tan(x) \rvert + C$$

For Ch. 13, you MUST remember:

- how to find +C.
- how to simplification and **u-substitution**.
- How to use **integration by parts**.

There are more methods, but you won't need to remember those until chapter 15.

Try again component-wise:

(HW 13.2 / 10)

Find the antiderivative of

$$\vec{\boldsymbol{r}}'(t) = \langle e^{3t}, t^4, t\sin(t)\rangle$$

with $\vec{\boldsymbol{r}}(0) = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$**.**